HARMONIC MORPHISMS BETWEEN RIEMANNIAN MANIFOLDS

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Abstract. *Harmonic morphisms* are mappings between Riemannian manifolds which preserve Laplace's equation. They can be characterized as harmonic maps which enjoy an extra property called *horizontal weak conformality* or *semiconformality.* We shall give a brief survey of the theory concentrating on (i) twistor methods, (ii) harmonic morphisms with one-dimensional fibres; in particular we shall outline the connections with two equations of Mathematical Physics: the *monopole equation* and the *Beltrami fields equation* of hydrodynamics.

Introduction

Harmonic morphisms are morphisms of Laplace's equation. This makes sense in the general setting of *Brelot harmonic space* [10]: these are topological spaces where the notion of 'harmonic function' is defined axiomatically; they include *Riemannian manifolds*; *Riemannian polyhedra* (J. Eells and B. Fuglede [13]); *metric graphs* (H. Urakawa [37]) and *Weyl spaces* (E. Loubeau and R. Pantilie [28, 33]). We shall study harmonic morphisms between Riemannian manifolds, starting with Euclidean spaces.

For mappings between Euclidean spaces, they can be characterized as maps which satisfy both Laplace's equation and some quadratic equations in the first derivatives. The idea of a harmonic morphism goes back to C.G.J. Jacobi (1848) who wanted to find new solutions to Laplace's equation on Euclidean 3-space $\mathbb{R}^3$ from old ones. In Section 1, we shall first of all discuss his ideas, and how they actually give all complex-valued harmonic morphisms on $\mathbb{R}^3$; we shall then see how this generalizes to $\mathbb{R}^4$. Finally we summarize what is known about harmonic morphisms between Euclidean spaces of higher dimensions.

In Section 2, we turn to the general case of harmonic morphisms between Riemannian manifolds, explaining the notions of *harmonic map* and *horizontal weak conformality.*

In Section 3, we summarize some of the important constructions of harmonic morphisms, namely twistorial constructions, and constructions for harmonic morphisms with one-dimensional fibres.

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Note that harmonic morphisms can also be characterized as *Brownian Path Preserving* maps, see [7], [15] (2.43). This was used by F. Duheille [11] to reprove the 'Bernstein'-type result (Theorem 1.6 below) of Baird and Wood.

For general information on harmonic morphisms see the book [6]. For a frequently updated bibliography of articles on harmonic morphisms, see *the Bibliography of Harmonic Morphisms*: `http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html.`

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1. Harmonic morphisms between Euclidean spaces

1.1. Jacobi's constructions. In order to find new harmonic functions on $\mathbb{R}^3$, Jacobi studied the following question [24] (for $m = 3$):

Let $\varphi : U \to \mathbb{C}$ be a C^2 function on an open subset of Euclidean m-space $\mathbb{R}^m$ which is harmonic *i.e., which satisfies* Laplace's equation:

$$\Delta\varphi \equiv \sum_{i=1}^{m} \frac{\partial^2 \varphi}{\partial x_i^2} = 0 \qquad \big(\boldsymbol{x} = (x_1, \dots, x_m) \in U\big)\,. \tag{1.1}$$

Under what further conditions on φ is the composition $f\circ\varphi$ harmonic for an arbitrary holomorphic (i.e. complex-analytic) map $f : V \to \mathbb{C}$ defined on an open subset of $\mathbb{C}$?

This is easy to answer using the chain rule; indeed

$$\frac{\partial}{\partial x_i}(f \circ \varphi) = \frac{\mathrm{d}f}{\mathrm{d}z}\frac{\partial \varphi}{\partial x_i} \qquad (i = 1, \dots, m)\,;$$

on differentiating this and summing we obtain

$$\Delta(f \circ \varphi) = \frac{\mathrm{d}f}{\mathrm{d}z}\,\Delta\varphi + \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} \sum_{i=1}^{m} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 . \tag{1.2}$$

Since we can find holomorphic maps $z \mapsto f(z)$with any prescribed values of their first and second derivatives $\mathrm{d}f/\mathrm{d}z$, $\mathrm{d}^2 f/\mathrm{d}z^2$ at a point, we deduce, as Jacobi did for $m = 3$ ([24] Section 5, p. 125), the following:

Proposition 1.1. *Let $\varphi : U \to \mathbb{C}$ be a harmonic function defined on an open subset of $\mathbb{R}^3$. Then $f \circ \varphi$ is harmonic for all holomorphic maps $f : V \to \mathbb{C}$ defined on open subsets of $\mathbb{C}$ if and only if φ satisfies the additional condition*

$$\sum_{i=1}^{m} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 = 0\,. \tag{1.3}$$

□

Definition 1.2. A continuous map $\varphi : U \to \mathbb{C}$ from an open subset of $\mathbb{R}^m$ is called a *harmonic morphism* if, whenever $h : V \to \mathbb{R}$ is a harmonic function on an open subset V of $\mathbb{C}$ with $\varphi^{-1}(V)$ non-empty, then $h \circ \varphi$ is harmonic.

Thus a harmonic morphism 'preserves' Laplace's equation. We shall extend this definition to maps between Riemannian manifolds in §2.

Example 1.3. Recall that a map is called *weakly conformal* if it is angle-preserving away from points where its differential vanishes, see §2.2 for more information on such mappings. A weakly conformal map between domains of the plane $\mathbb{R}^2 = \mathbb{C}$ is precisely the same as a holomorphic or antiholomorphic map. It is classsical that such a map is a harmonic morphism.

Note that a complex-valued harmonic morphism $\varphi : U \to \mathbb{C}$ defined on an open subset U of $\mathbb{R}^m$ remains a harmonic morphism when *precomposed* with an *isometry* $U' \to U$ (U' open in $\mathbb{R}^m$) or *postcomposed* with a complex-valued *weakly conformal* (i.e., holomorphic or antiholomorphic) map $V \to \mathbb{C}$ (V open in $\mathbb{C}$). The conformal invariance means that we can consider harmonic morphisms with values in a *Riemann surface*, for example, the *Riemann sphere* is the 2-sphere S^2 identified conformally with the extended complex plane $\mathbb{C}\cup\{\infty\}$ by *stereographic projection*:

$$S^2 \ni (x_1, x_2, x_3) \mapsto \frac{1}{1+x_1}\, x_2 + \mathrm{i}x_3 \in \mathbb{C} \cup \{\infty\}\,. \tag{1.4}$$

Proposition 1.1 leads to the following characterization of complex-valued harmonic morphism on domains of $\mathbb{R}^m$:

Proposition 1.4. *Let $\varphi : U \to \mathbb{C}$ be a continuous map from an open subset of $\mathbb{R}^m$. Then φ is a harmonic morphism if and only if it is smooth and satisfies equations* (1.1) *and* (1.3).

Proof. First note that any harmonic morphism is smooth (i.e., C^∞)—in fact it is real-analytic—since, when we set h equal to the harmonic function $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ which gives the real part of a complex number, the composition $h\circ\varphi$ gives the real part of φ. By definition of harmonic morphism, this is a harmonic function and so is smooth. Similary, the imaginary part of φ is smooth. Then, since any harmonic function h is locally the real part of a holomorphic function f, the proposition follows quickly from the chain rule (1.2). ☐

So to find harmonic morphisms, we must solve the pair (1.1, 1.3). This pair of equations is easy to solve when $m = 2$: then (1.3) is equivalent to

$$\frac{\partial\varphi}{\partial x_2} = \pm\,\mathrm{i}\frac{\partial\varphi}{\partial x_1}\,;$$

these are the Cauchy–Riemann, or conjugate Cauchy–Riemann equations. They imply (1.1), so that *a map* $\varphi : U \to \mathbb{C}$ *defined on an open subset of* $\mathbb{R}^2 = \mathbb{C}$ *is a harmonic morphism if and only if it is holomorphic or antiholomorphic, equivalently it is weakly conformal.* That weakly conformal maps between plane domains preserve Laplace's equation is, of course, classical; we have just shown that these are the *only* maps between plane domains with that property.

To solve the pair (1.1, 1.3) for $m = 3$, Jacobi came up with the idea of implicitly defining them using 'known' harmonic morphisms, as follows.

Proposition 1.5. *Let* $\Pi : \mathbb{R}^m \times \mathbb{C} \supseteq A \to \mathbb{C}, \ (\boldsymbol{x}, z) \mapsto w$ *be*

(i) *a harmonic morphism in its first argument* $\boldsymbol{x}$*, i.e.,*

$$\sum_{i=1}^{m} \frac{\partial^2 \Pi}{\partial {x_i}^2} = 0 \quad \text{and} \quad \sum_{i=1}^{m} \left(\frac{\partial \Pi}{\partial x_i} \right)^2 = 0\,;$$

(ii) *holomorphic in its second argument* z*.*

Let $w_0 \in \mathbb{C}$ *and suppose that* $\mathrm{d}\Pi \neq 0$ *on* $\Pi^{-1}(w_0)$*.*

Then any smooth local solution

$$\varphi : U \to \mathbb{C}, \quad z = \varphi(\boldsymbol{x})$$

defined on an open subset U *of* $\mathbb{R}^m$ *to the equation*

$$\Pi(\boldsymbol{x}, z) = w_0$$

is a harmonic morphism.

Proof. This is a simple exercise using the chain rule. □

1.2. **Complex-valued harmonic morphisms on** $\mathbb{R}^3$**.** To apply Proposition 1.5, set

$$\Pi(\boldsymbol{x}, z) = a_1(z)x_1 + a_2(z)x_2 + a_3(z)x_3 - b(z)\,.$$

Since linear, Π is *harmonic in* $\boldsymbol{x}$: $\displaystyle\sum_{i=1}^{3} \frac{\partial^2 \Pi}{\partial {x_i}^2} = 0$. It is *horizontally weakly conformal in* $\boldsymbol{x}$ if and only if $\sum_{i=1}^{3} {a_i}^2 \equiv 0$. It is *holomorphic in* z if and only if the a_i and b are holomorphic in z. Triples $\boldsymbol{a}(z) = (a_1(z), a_2(z), a_3(z))$ of holomorphic functions satisfying $\sum_{i=1}^{3} {a_i}^2 \equiv 0$ are given by

$$\boldsymbol{a}(z) = \frac{1}{2h(z)} \big(-2g(z),\, 1 - g(z)^2,\, \mathrm{i}(1 + g(z)^2) \big)\,.$$

for holomorphic functions g and h. We have thus proved part (i) of the following result.

Theorem 1.6. (Weierstrass formula for harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$) *Let* $g, h : B \to \mathbb{C}$ *be holomorphic functions defined on a domain of* $\mathbb{C}$ *(or on a Riemann surface).*

Let $\varphi : U \to \mathbb{C}, \quad z = \varphi(x_1, x_2, x_3)$ *be a smooth solution to equation*

$$-2g(z)x_1 + \big(1 - g(z)^2\big)x_2 + \mathrm{i}\big(1 + g(z)^2\big)x_3 = 2h(z) \tag{1.5}$$

defined on an open subset U *of* $\mathbb{R}^3$. *Then*

(i) φ *is a harmonic morphism;*

(ii) (P. Baird and J. C. Wood [3]) *all harmonic morphisms from open subsets of* $\mathbb{R}^3$ *to* $\mathbb{C}$ *or to Riemann surfaces are given this way locally, up to composition with isometries of the domain and weakly conformal maps on the codomain.*

(iii) (P. Baird and J. C. Wood [3]) *The only harmonic morphism defined globally on* $\mathbb{R}^3$ *is* orthogonal projection $z = x_2 + \mathrm{i}x_3$ *up to composition with isometries of the domain and weakly conformal maps on the codomain.*

To prove (ii), one first of all proves that any harmonic morphism can be factorized on suitable domains into the composition of a submersive harmonic morphism and a weakly conformal map, so that we can assume that our harmonic morphism is submersive. Then, by a direct argument or by Theorem 3.3 below, we see that any harmonic morphism from an open subset of $\mathbb{R}^3$ to $\mathbb{C}$ to a Riemann surface has fibres which are portions of straight lines. The horizontal conformality then ensures that these lines vary in a holomorphic fashion, and so are determined by two holomorphic functions g and h.

To prove (iii), we show that some of the lines given by g and h will intersect somewhere unless g is constant, i.e., all lines are parallel.

Example 1.7. (i) Setting $g = 0$, $h(z) = \frac{1}{2}z$ gives the orthogonal projection of part (iii). Note that $g = 0$ implies that the fibres are portions of straight lines parallel to the x_1-axis.

(ii) On identifying the extended complex plane with the Riemann sphere conformally by stereographic projection (1.4), $g(z) = z$, $h = 0$ gives the *radial projection* $\mathbb{R}^3 \setminus \{\mathbf{0}\} \to S^2$, $\boldsymbol{x} \mapsto \boldsymbol{x}/|\boldsymbol{x}|$. Note that $h = 0$ implies that the fibres are portions of straight lines through the origin.

See [6] Chapter 1 for more examples.

1.3. **Complex-valued harmonic morphisms on $\mathbb{R}^4$.**

To find complex-valued harmonic morphisms from domains of $\mathbb{R}^4$, first note that any complex-valued holomorphic (or antiholomorphic) function defined on a domain of $\mathbb{R}^4 = \mathbb{C}^2$ is a harmonic morphism. Indeed, it is easily checked that the Cauchy–Riemann equations imply equations (1.1) and (1.3). More general solutions are provided by a generalization of the Weierstrass-type construction for $\mathbb{R}^3$, as follows.

Let f be a holomorphic function of three complex variables with $\mathrm{d}f$ nowhere zero, and μ a holomorphic function of one complex variable. Writing $\boldsymbol{x} = (q_1, q_2) \in \mathbb{C}^2 = \mathbb{R}^4$, set

$$\Pi(\boldsymbol{x}, z) = f(z,\, q_1 - \mu(z)\overline{q_2},\, q_2 + \mu(z)\overline{q_1})\,.$$

Then Π is a harmonic morphism in $\boldsymbol{x}$ (check!) and holomorphic in z. Proposition 1.5 then immediately gives part (i) of the following result.

Theorem 1.8. *Let* $\varphi : \mathbb{R}^4 \supseteq U \to \mathbb{C}\,,\ z = \varphi(\boldsymbol{x})$ *be a smooth submersive solution to the equation*

$$\Pi(\boldsymbol{x}, z) = 0\,. \tag{1.6}$$

Then (i) φ *is a (submersive) harmonic morphism;*

(ii) (J. C. Wood [39]) *each submersive harmonic morphism from an open subset of* $\mathbb{R}^4$ *to* $\mathbb{C}$ *or to a Riemann surface is given this way locally, up to composition with isometries of* $\mathbb{R}^4$ *and conformal mappings of the codomain.*

(iii) (J. C. Wood [39]) *any submersive harmonic morphism from* $\mathbb{R}^4$ *is holomorphic, up to precomposition with an isometry of* $\mathbb{R}^4$*.*

In fact, results of M. Ville [38], allow us to prove (ii) and (iii) without the 'submersive' condition. We remark that the construction of Theorem 1.6 can be seen as a 'reduction' to three dimensions of the construction in this theorem, see [6] Example 8.5.4.

Example 1.9. Set $f(z, w_1, w_2) = w_1$ and $\mu(z) = z$. Then equation (1.6) reads

$$q_1 - z\overline{q_2} = 0,$$

with solution

$$z = \varphi(q_1, q_2) = q_1/\overline{q_2}\,.$$

This gives the harmonic morphism φ. With the notion of harmonic morphism between general Riemannian manifolds (see §2), we can interpret this as the composition of three harmonic morphisms:

$$\mathbb{R}^4 \setminus \{\mathbf{0}\} \xrightarrow{\text{radial}} S^3 \xrightarrow{\overline{\text{Hopf}}} S^2 \xrightarrow{\text{stereo}} \mathbb{C} \cup \{\infty\}$$

Here, 'radial' denotes radial projection (Example 2.7), '$\overline{\text{Hopf}}$' denotes the standard Hopf map (see Example 3.11) up to isometries. and 'stereo' denotes stereographic projection (1.4). Precomposing this map with the isometry which replaces $\overline{q_2}$ by q_2 gives the harmonic morphism:

$$\begin{array}{ccccc} \mathbb{C}^2 \setminus \{\mathbf{0}\} & \xrightarrow{\text{standard projn}} & \mathbb{C}P^1 & \xrightarrow{\text{standard identification}} & \mathbb{C} \cup \{\infty\} \\ (q_1, q_2) & \mapsto & [q_1, q_2] & \mapsto & q_1/q_2 \end{array}$$

If we chose a more complicated formula for f, e.g., quadratic in (w_1, w_2), we may get several solutions to (1.6), which we can interpret as a *multivalued harmonic morphism* ([21], see also [6] Chapter 9).

The constructions above can be generalized to find complex-valued harmonic morphisms from higher-dimensional Euclidean spaces, see [4, 5, 40]; however, it is not known whether we obtain *all* harmonic morphisms by such methods.

1.4. **Harmonic morphisms between Euclidean spaces of arbitrary dimensions.** A chain rule argument similar to that used in §1.1 shows:

Proposition 1.10. *A smooth map $\varphi : U \to \mathbb{R}^n$ defined on an open subset U of $\mathbb{R}^m$ is a harmonic morphism if and only if it is* harmonic*:*

$$\Delta\varphi = 0\,. \tag{1.7}$$

and horizontally weakly conformal *in the following sense:*

$$\sum_{i=1}^{m} \frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_i} = 0 \quad \textit{and} \quad \sum_{i=1}^{m}\left(\frac{\partial\varphi^\alpha}{\partial x_i}\right)^2 = \sum_{i=1}^{m}\left(\frac{\partial\varphi^\beta}{\partial x_i}\right)^2 \tag{1.8}$$

$$(\alpha, \beta \in \{1, \dots, m\},\ \alpha \neq \beta).$$

Geometrically, (1.8) this says that *the gradients of the components φ^α of φ are orthogonal and of the same length.*

It is an open problem to find all harmonic morphisms between (open subsets of) Euclidean spaces. The following is known: (i) (R. Ababou, P. Baird, J. Brossard [1]) *for maps given by polynomials, horizontal weak conformality* (1.8) *implies harmonicity* (1.7); (ii) (Y.-L. Ou and J. C. Wood [30], Y. L. Ou [29]) *All* quadratic *harmonic morphisms are given by* orthogonal multiplications *or, equivalently,* Clifford systems. See [6] Chapter 5 for more information.

Some value-distribution results can be found using the Brownian path-preserving characterization of harmonic morphisms mentioned in the Introduction. For example, F. Duheille [12] shows that, under some general assumptions, *a non-constant harmonic morphism from $\mathbb{R}^m$ to $\mathbb{R}^3$ ($m > 3$) cannot avoid three concurrent half-lines.*

2. Harmonic morphisms between Riemannian manifolds

In the rest of the paper, we shall discuss harmonic morphisms between more general Riemanian manifolds.

2.1. **Harmonic maps.** To study harmonic morphisms between Riemannian manifolds, we need to recall the notion of *harmonic map.*

From now on, (M, g) and (N, h) will denote (smooth) Riemannian manifolds or arbitrary (finite) dimensions m and n, respectively. We define harmonic maps as the solution to a variational problem for maps from (M, g) to (N, h):

Let $\varphi : (M, g) \to (N, h)$ be a smooth map. For simplicity, we shall assume that M is compact. The *energy* or *Dirichlet integral* of φ is the non-negative number:

$$E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2\,\omega_g \tag{2.1}$$

where ω_g is the volume measure on M defined by the metric g, and, for any $x \in M$, $|\mathrm{d}\varphi_x|$ is the Hilbert–Schmidt norm of $\mathrm{d}\varphi_x$ defined by

$$|\mathrm{d}\varphi_x|^2 = \sum_{i=1}^{m} h\big(\mathrm{d}\varphi_x(e_i), \mathrm{d}\varphi_x(e_i)\big) \tag{2.2}$$

where $\{e_i\}$ is an orthonormal basis for T_xM. In local coordinates $(x^1, \dots, x^m)$ on M, $(u^1, \dots, u^n)$ on N,

$$|\mathrm{d}\varphi|^2 = g^{ij} h_{\alpha\beta}\, \varphi_i^\alpha \varphi_j^\beta \quad \text{and} \quad \omega_g = |g|\, \mathrm{d}x^1 \cdots \mathrm{d}x^m\,; \tag{2.3}$$

here φ_i^α denotes the partial derivative $\partial\varphi^\alpha/\partial x^i = \partial u^\alpha/\partial x^i$; also (g_{ij}) denotes the metric tensor on M with respect to the chosen local coordinates, $|g|$ denotes its determinant and (g^{ij}) its inverse, and $(h_{\alpha\beta})$ denotes the metric tensor on N.

By a *smooth (one-parameter) variation* $\{\varphi_t\}$ *of* φ we mean a smooth map $M \times (-\epsilon, \epsilon) \to N$, $(x, t) \mapsto \varphi_t(x)$ where $\epsilon > 0$ and $\varphi_0 = \varphi$. A smooth map $\varphi : (M, g) \to (N, h)$ is called *harmonic* if it is an *extremal* of the energy integral, i.e., for all smooth one-parameter variations $\{\varphi_t\}$ of φ, the *first variation* $\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t)\big|_{t=0}$ is zero. We compute ([17, 15, 36, 41]):

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t)\Big|_{t=0} = -\int_M \big\langle \tau(\varphi), v \big\rangle\, \omega_g\,. \tag{2.4}$$

Here v denotes the *variation vector field of* $\{\varphi_t\}$ defined by $v = \partial\varphi_t/\partial t|_{t=0}$, and $\tau(\varphi)$ is called the *tension field of* φ. These are both *vector fields along* φ, i.e., sections of the pull-back bundle $\varphi^{-1}TN \to M$; thus, for each $x \in M$, $v(x)$ and $\tau(\varphi)(x) \in T_{\varphi(x)}N$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product on $\varphi^{-1}TN$ induced from the metric on N, thus $\big\langle \tau(\varphi), v \big\rangle_x = h\big(\tau(\varphi)(x), v(x)\big)$ $(x \in M)$. The tension field $\tau(\varphi)$ is defined by

$$\tau(\varphi) = \mathrm{trace}\nabla\mathrm{d}\varphi = \sum_{i=1}^{m} \nabla\mathrm{d}\varphi(e_i, e_i) = \sum_{i=1}^{m} \big\{ \nabla^\varphi_{e_i}\big(\mathrm{d}\varphi(e_i)\big) - \mathrm{d}\varphi(\nabla^M_{e_i} e_i) \big\}\,.$$

Here ∇^M denotes the Levi-Civita connection on TM, ∇^φ the pull-back of the Levi-Civita connection ∇^N on N to the bundle $\varphi^{-1}TN$, and ∇ the connection on $T^*M \otimes \varphi^{-1}TN$ induced from these connections. In local coordinates,

$$\begin{aligned} \tau(\varphi)^\gamma &= g^{ij} \left(\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \varphi^\gamma}{\partial x^k} + L^\gamma_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j} \right) \\ &= \Delta^M \varphi^\gamma + g(\mathrm{grad}\varphi^\alpha,\, \mathrm{grad}\varphi^\beta)\, L^\gamma_{\alpha\beta}\,. \end{aligned}$$

Here, Γ^k_{ij} and $L^\gamma_{\alpha\beta}$ denote the Christoffel symbols on M and N, respectively, and Δ^M denotes the Laplace–Beltrami operator on functions $f : M \to \mathbb{R}$ given by:

(2.5) $$\Delta^M f = \mathrm{trace}\nabla \mathrm{d}f = \sum_{i=1}^{m} \big\{ e_i\big(e_i(f)\big) - \big(\nabla^M_{e_i} e_i\big) f \big\}\,,$$

or, in local coordinates,

(2.6) $$\Delta^M f = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\, g^{ij}\frac{\partial f}{\partial x^j}\right) = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right).$$

Note that $\tau(\varphi)$ can be interpreted as the negative of the gradient of the energy functional E (viewed as a real-valued function on a suitable space of mappings from M to N), i.e., it points in the direction in which E decreases most rapidly. In local coordinates, the *harmonic equation* $\tau(\varphi) = 0$ is a *semilinear* system of elliptic partial differential equations; there are general existence theorems for maps with small range, or into manifolds of non-positive curvature, but not for maps into more general manifolds. See [14, 16] for more information on harmonic maps.

2.2. **Weak conformality and horizontal weak conformality.** A notion important in the theory of harmonic maps is that of 'weak conformality'. We write that in such a way as to invite comparision with the dual notion of 'horizontal weak conformality' defined below.

Definition 2.1. A smooth map $\varphi : (M,g) \to (N,h)$ is called *weakly conformal* if, for each $p \in M$, *either*,

(i) $\mathrm{d}\varphi_p = 0$, in which case we call p a *branch point*, *or*,

(ii) $\mathrm{d}\varphi_p$ maps the tangent space T_pM conformally into $T_{\varphi(p)}N$, i.e., $\mathrm{d}\varphi_p$ is injective and there exists a number $\lambda(p) \neq 0$ such that

$$h\big(\mathrm{d}\varphi_p(X), \mathrm{d}\varphi_p(Y)\big) = \lambda(p)^2 g(X,Y) \qquad (X, Y \in T_pM)\,,$$

in which case we call p a *regular point.*

By a *conformal map* we mean a weakly conformal map with no branch points.

Set $\lambda = 0$ at branch points; then $\lambda : M \to [0,\infty)$ is a continuous function called the *conformality factor* of φ; note that λ^2 is smooth since it equals $|\mathrm{d}\varphi|^2/m$.

At its regular points a weakly conformal map *preserves angles.* There are many other ways to write the definition; we shall give three of them. For each $p \in M$, let $\mathrm{d}\varphi_p^* : T_{\varphi(p)}N \to T_pM$ denote the *adjoint* of $\mathrm{d}\varphi_p$ characterized by

$$g(X, \mathrm{d}\varphi^*_p(Y)) = h(\mathrm{d}\varphi_p(X), Y) \qquad (X \in T_pM,\ Y \in T_{\varphi(p)}N)\,.$$

Then we have:

Lemma 2.2. *Let $\varphi : (M,g) \to (N,h)$ be a smooth map. Then the following conditions are equivalent.*

(i) *φ is weakly conformal with conformality factor λ;*

(ii) *$h(\mathrm{d}\varphi_p(X), \mathrm{d}\varphi_p(Y)) = \lambda(p)^2 g(X,Y)$ for all $p \in M$, $X, Y \in T_pM$;*

(iii) *at each point* $p \in M$ *we have* $\mathrm{d}\varphi_p^* \circ \mathrm{d}\varphi_p = \lambda^2 \, \mathrm{Id}_{T_pM}$ *;*

(iv) *in local coordinates,* $h_{\alpha\beta}\varphi_i^\alpha\varphi_j^\beta = \lambda^2 \, g_{ij}$ *.*

See [6] §2.3 for other equivalent conditions and more information on weakly conformal maps.

We next discuss the dual notion of 'horizontal weak conformality', which is important in the theory of harmonic morphisms.

Definition 2.3. A smooth map $\varphi : (M, g) \to (N, h)$ is called *horizontally weakly conformal* (or *semiconformal*) if, for each $p \in M$, *either*,

(i) $\mathrm{d}\varphi_p = 0$, in which case we call p a *critical point*, or,

(ii) $\mathrm{d}\varphi_p$ maps the horizontal space $\mathcal{H}_p = \{\ker(\mathrm{d}\varphi_p)\}^\perp$ conformally onto $T_{\varphi(p)}N$, i.e., $\mathrm{d}\varphi_p$ is surjective and there exists a number $\lambda(p) \neq 0$ such that

$$h\big(\mathrm{d}\varphi_p(X), \mathrm{d}\varphi_p(Y)\big) = \lambda(p)^2 \, g(X, Y) \qquad (X, Y \in \mathcal{H}_p)\,,$$

in which case we call p a *regular point*.

By a *horizontally conformal map* or *(horizontally) conformal submersion* we mean a horizontally weakly conformal map with no branch points.

Set $\lambda = 0$ at critical points; then $\lambda : M \to [0, \infty)$ is a continuous function called the *dilation* of φ; note that λ^2 is smooth, since it equals $|\mathrm{d}\varphi|^2/n$.

At regular points, a horizontally weakly conformal map preserves angles on the horizontal space. There are many other ways to write the definition, see [6] §2.4. We give three of them; again, for each $p \in M$, $\mathrm{d}\varphi_p^* : TN \to TM$ denotes the adjoint of $\mathrm{d}\varphi_p$; compare with Lemma 2.2.

Lemma 2.4. *Let* $\varphi : (M, g) \to (N, h)$ *be a smooth map. Then the following conditions are equivalent.*

(i) φ *is horizontally weakly conformal with dilation* λ *;*

(ii) $g(\mathrm{d}\varphi_p^*(X), \mathrm{d}\varphi_p^*(Y)) = \lambda^2(p) \, h(X, Y)$ *for all* $p \in M$, $X, Y \in T_{\varphi(p)}N$ *;*

(iii) *at each point* $p \in M$ *we have* $\mathrm{d}\varphi_p \circ \mathrm{d}\varphi_p^* = \lambda^2 \, \mathrm{Id}_{T_{\varphi(p)}N}$ *;*

(iv) *in any local coordinates,* $g^{ij}\varphi_i^\alpha\varphi_j^\beta = \lambda^2 h^{\alpha\beta}$ *.*

2.3. **Harmonic morphisms.** We can immediately generalize Definition 1.2 to maps between (smooth) Riemannian manifolds (M, g) and (N, h).

Definition 2.5. A continuous map $\varphi : M \to N$ is called a *harmonic morphism* if, for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty, the composition $f \circ \varphi$ is harmonic on $\varphi^{-1}(V)$.

It follows that φ is smooth (and real analytic if (M, g) and (N, h) are real analytic), since harmonic functions have that property (see, for example, [6] Proposition 4.3.1).

B. Fuglede [18] and T. Ishihara [23] independently gave the following characterization.

Theorem 2.6. *A smooth map $\varphi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is both* harmonic *and* horizontally weakly conformal*.*

This is proved by calculating the tension field of the composition of φ with a harmonic function $f : N \supset V \to \mathbb{R}$, and showing that there is a harmonic function with any prescribed (traceless) 2-jet; see [6] §4.2.

Example 2.7. (i) The *Hopf maps* from $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$, $S^{2n+1} \to \mathbb{C}P^n$, $S^{4n+3} \to \mathbb{H}P^n$ are all *Riemannian submersions* up to scale, i.e., horizontally conformal submersions with λ constant. It can be checked that they are also harmonic maps; hence they are harmonic morphisms.

(ii) *Radial projection.* It can easily be checked that, for any $m \in \{1, 2, \ldots\}$, the map

$$\mathbb{R}^m \setminus \{\mathbf{0}\} \to S^{m-1}, \quad \boldsymbol{x} \mapsto \boldsymbol{x}/|\boldsymbol{x}|$$

is (i) a horizontally conformal submersion with dilation $\lambda(\boldsymbol{x}) = 1/|\boldsymbol{x}|$, (ii) harmonic. Hence it is a harmonic morphism.

See Example 3.4 below for a simpler argument for these examples based on minimality of the fibres.

(iii) [J.Y. Chen, 1994] *Stable harmonic maps* from a compact Riemannian manifold to S^2 are all harmonic morphisms. This is shown by calculating the second variation and showing that its non-negativity forces the map to be horizontally weakly conformal.

3. Construction and classification of harmonic morphisms.

3.1. **Twistor constructions.** It is not hard to see that the harmonic morphisms constructed in Theorem 1.8 are holomorphic with respect to the almost Hermitian structure J on U which at any point $\boldsymbol{x} = (q_1, q_2)$ has $(0,1)$-tangent space spanned by

$$\frac{\partial}{\partial \overline{q}_1} - \mu(\varphi(\boldsymbol{x}))\frac{\partial}{\partial q_2} \quad \text{and} \quad \frac{\partial}{\partial \overline{q}_2} + \mu(\varphi(\boldsymbol{x}))\frac{\partial}{\partial q_1}$$

Clearly, J is parallel (constant) along the fibres; it is easy to see ([6] Proposition 7.9.1) that this is equivalent to integrability of J. This is generalized by the following result. Say that a surface is *superminimal* with respect to an almost Hermitian structure J if its tangent space is closed under J and J is parallel along the surface.

Theorem 3.1. *Let $\varphi : M^4 \to N^2$ be a non-constant harmonic morphism from an orientable Einstein 4-manifold to a Riemann surface. Then φ is holomorphic with respect to some integrable Hermitian structure J on M^4 and has superminimal fibres with respect to J.*

This was proved by the author in [39] for submersive harmonic morphisms; the submersivity condition was removed by M. Ville [38]. For a converse, see [6] §7.10.

Call a map *twistorial* if it is covered by a holomorphic map between suitable fibre bundles, for precise definitions see [35], and in the setting of harmonic morphisms between Weyl spaces, [28, 33]. Then Theorem 3.1 says that *any harmonic morphism from an orientable Einstein* 4-*manifold to a Riemann surface is twistorial*; see the papers just cited for more theorems of this type.

See [4, 5] for constructions of harmonic morphisms from certain higher-dimensional manifolds inspired by twistorial methods, and [20] for a different method of construction of complex-valued harmonic morphisms from symmetric spaces.

3.2. **The fundamental equation.**

Proposition 3.2. (P. Baird and J. Eells [2]) *Let* $\varphi : M^m \to N^n$ *be horizontally weakly conformal with dilation* λ*. Then, at a regular point,*

$$\tau(\varphi) = \mathrm{d}\varphi\big\{-(n-2)\mathrm{grad}\,\ln\lambda - (m-n)\mu^{\mathcal{V}}\big\}\,. \tag{3.1}$$

Here $\mu^{\mathcal{V}}$ denotes the mean curvature of the fibres. Thus φ is harmonic, and so a harmonic morphism, if and ony if, at regular points,

$$(n-2)\mathcal{H}(\mathrm{grad}\,\ln\lambda) + (m-n)\mu^{\mathcal{V}} = 0\,.$$

Theorem 3.3. (P. Baird and J. Eells [2]) *If* $n = 2$*,* or $\mathrm{grad}\,\lambda$ *is vertical at regular points, a horizontally weakly conformal map is harmonic, and so is a harmonic morphism, if and only if its fibres are minimal at regular points.*

A horizontally weakly conformal map with $\mathrm{grad}\,\lambda$ vertical is called a *horizontally homothetic map.* For example, a Riemannian submersion is horizontally homothetic.

Example 3.4. (i) A *Riemannian submersion* is harmonic, and so a harmonic morphism, if and only if its fibres are minimal. The Hopf maps in Example 2.7(i) have minimal (in fact, totally geodesic) fibres, and so are harmonic morphisms by Proposition 3.2.

(ii) The natural projection of a *warped product* $F \times_{f^2} N \to N$ onto its second factor is a horizontally homothetic map with totally geodesic fibres and integrable horizontal distribution. In particular, it is a harmonic morphism. Conversely, a horizontally homothetic map with totally geodesic fibres and integrable horizontal distribution is locally the projection of a warped product. We call such maps *harmonic morphisms of warped product type* (or *umbilic harmonic morphisms* [8]). The radial projections of Example 2.7(ii) are such maps.

Given a submersion $\varphi : M \to N$ between Riemannian manifolds, the connected components of its fibres form a smooth *foliation* by submanifolds of dimension $q = \dim M - \dim N$. A foliation is called *conformal* (resp. *Riemannian*) if it is locally given by a horizontally conformal submersion (resp. a Riemannian submersion). Given a conformal foliation $\mathcal{F}$, we may ask for conditions under which it is given locally by harmonic morphisms; in this case we shall say that $\mathcal{F}$ *produces harmonic morphisms*. Let $\mu^{\mathcal{V}}$ denote the mean curvature of the leaves of $\mathcal{F}$ and $\mu^{\mathcal{H}}$ the mean curvature of the *horizontal distribution* $\mathcal{H}$ (i.e., the distribution of subspaces orthogonal to the leaves). It is easy to see that, for the foliation given by a horizontally conformal submersion of dilation λ, the vector field $\mu^{\mathcal{H}}$ is equal to the vertical component of grad $\ln \lambda$. Then the fundamental equation quickly gives:

Proposition 3.5. (i) *If $n = 2$, a conformal foliation produces harmonic morphisms if and only if its leaves are minimal, i.e., $\mu^{\mathcal{V}} = 0$.*

(ii) (R. L. Bryant [8]) *If $n \neq 2$, a conformal foliation produces harmonic morphisms if and only if*

$$W = (n-2)\mu^{\mathcal{H}} - (m-n)\mu^{\mathcal{V}}$$

is locally a gradient field, equivalently, $W^\flat$ is closed.

Here $W^\flat$ denotes the 1-form asociated to W by the musical isomorphism $\flat : TM \to T^*M$ defined by the metric on M.

The fundamental equation (3.1) for a horizontally weakly conformal map admits an interesting interpretation. First, an easy calculation gives the following interpretation of the mean curvature of the fibres as a Lie derivative of the volume form.

Lemma 3.6. *Let $\mathcal{F}$ be a foliation of dimension q on a Riemannian manifold and let $v^{\mathcal{V}}$ be the vertical form on M which gives the volume form on each leaf (with respect to some local orientation). Then for any horizontal vector field X,*

$$\mathcal{V}^*\big(\mathcal{L}_X(v^{\mathcal{V}})\big) = -q\, g(\mu^{\mathcal{V}}, X) v^{\mathcal{V}}\,.$$

Here $\mathcal{V}^*$ denotes 'vertical part'. Now, for any $p \in M$, let $v^\wedge$ denote the lift of a vector $v \in T_{\varphi(p)}N$ to $\mathcal{H}_p$ using the inverse of $\mathrm{d}\varphi_p$. Using the last lemma, we see that the fundamental equation (3.1) can be written as

$$g(\tau(\varphi)^\wedge, X)\, v^{\mathcal{V}} = \lambda^{n-2}\, \mathcal{V}^*\big(\mathcal{L}_X(\lambda^{2-n} v^{\mathcal{V}})\big)\,. \tag{3.2}$$

We remark that we can interpret $\mathcal{V}^* \circ \mathcal{L}_X$ as a *Bott partial connection* on the vertical bundle $\mathcal{V}$, so that *a horizontally weakly conformal map φ is a harmonic morphism if and only if, at regular points, the form $\lambda^{2-n} v^{\mathcal{V}}$ is parallel in all horizontal directions with respect to the Bott partial connection*. See [6] §4.6 for this and related matters.

3.3. **Harmonic morphisms with one-dimensional fibres.** In the case of one-dimensional fibres, the fundamental equation (3.2) for a horizontally weakly conformal map can be rewritten as follows. Let $V = \lambda^{n-2} U$ where U is the unit positive tangent vector field to the fibres (with respect to some local orientation), and let θ be the dual vertical 1-form, i.e., $\theta(V) = 1$ and $\ker\theta = \mathcal{H}$. Then $\theta = \lambda^{2-n}U^\flat$ and we have that, at all regular points, for each horizontal vector field X,

$$g(\tau(\varphi)^\wedge, X) = (\mathcal{L}_X\theta)(V) = -\mathrm{d}\theta(V, X) = -(\mathcal{L}_V\theta)(X)\,. \tag{3.3}$$

We thus deduce

Proposition 3.7. *A horizontally weaky conformal map φ is a harmonic morphism if and only if, at all regular points, $\mathcal{L}_V(\theta) = 0$, equivalently, its horizontal distribution $\mathcal{H}$ is invariant under the flow of V.*

Thus at regular points, a harmonic morphism is locally a principal bundle with group $\mathbb{R}$ or S^1 and connection form θ. Furthermore, we can say that the metric g is of the form

$$g = \lambda^{-2}\,\varphi^*(h) + \lambda^{2n-4}\,\theta^2. \tag{3.4}$$

Indeed the first term expresses the horizontal conformality with dilation λ, and the second term is the square of the volume form on the fibres. We are thus led to the following statement.

Theorem 3.8. (R. L. Bryant [8]) *Let (M^{n+1}, N^n, S^1) be a principal bundle with projection $\varphi : M^{n+1} \to N^n$ and endowed with a principal connection $\mathcal{H} \subseteq TM$. Let h be a Riemannian metric on N^n and λ a smooth positive function on M^{n+1}.*

Define a Riemannian metric on M^{n+1} by (3.4) *where θ is the connection form of $\mathcal{H}$. Then $\varphi : (M^{n+1}, g) \to (N^n, h)$ is a submersive harmonic morphism.*

Conversely, any submersive harmonic morphism with one-dimensional fibres is locally *of this form, up to isometries.*

See [31] Theorem 2.9 (or [6] Theorem 12.2.6) for a proof of Theorem 3.8 and a more explicit version of the converse.

We next determine under what conditions the fundamental vector field V is a Killing vector field, i.e., $\mathcal{L}_V g = 0$. To do this, note that the expressions in (3.3) are all equal to $-\lambda^{4-2n}(\mathcal{L}_V g)(V, X)$, and that, for any horizontally conformal submersion the other two components of $\mathcal{L}_V g$ are given by

$$(\mathcal{L}_V g)(V, V) = V(\lambda^{2n-4}) \quad\text{and}\quad (\mathcal{L}_V g)(X, Y) = -V(\ln\lambda^2)\,g(X, Y)\,.$$

We deduce the following result, essentially due to R. L. Bryant [8].

Proposition 3.9. ([6] Proposition 12.3.1) *Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ $(n \geq 1)$ be a horizontally conformal submersion with one-dimensional fibres. Denote its dilation by λ and its fundamental vertical vector by V. Then V is Killing if and only if φ is a harmonic morphism with* $\operatorname{grad}\lambda$ *horizontal, i.e., with dilation constant along the fibre components.*

Call such maps *harmonic morphisms of Killing type.*

The foliation given by a horizontally conformal submersion with $\operatorname{grad}\lambda$ horizontal is a Riemannian foliation. We can give a version of the last result for such foliations.

Corollary 3.10. (R. L. Bryant [8], p. 234) *Let $\mathcal{F}$ be a one-dimensional Riemannian foliation on a Riemannian manifold M^{n+1} $(n \neq 2)$. Then $\mathcal{F}$ produces harmonic morphisms if and only if its leaves are tangent to (locally defined) Killing fields.*

Example 3.11. (The Hopf polynomial map) The map $\mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$\mathbb{C}^2 \ni (x,y) \mapsto \big(|x|^2 - |y|^2, 2\overline{x}y\big) \in \mathbb{R} \times \mathbb{C}$$

is a harmonic morphism of Killing type. It has fibres spanned by the Killing field corresponding to the isometric action:

$$t \cdot (x,y) = (\mathrm{e}^{\mathrm{i}t}x\,,\, \mathrm{e}^{\mathrm{i}t}y) \qquad (t \in \mathbb{R})\,.$$

It restricts to the Hopf map $S^3 \to S^2$.

3.4. **Connections with two equations of Mathematical Physics.** We show that harmonic morphisms are related to two equations of Mathematical physics: the *monopole equation* and the *Beltrami fields equation.*

In the following result, the construction in part (ii) (resp. (iii)) is due to P. E. Jones and K. P. Tod [25] (resp. G. W. Gibbons and S. W. Hawking [19]). See also [26, 27].

Theorem 3.12. *Let N^3 be an oriented constant curvature 3-manifold, set $M^4 = \mathbb{R} \times N^3$, and let $\varphi : M^4 \to N^3$ be projection onto the second factor. Define a Riemannian metric g on $M^4 = \mathbb{R} \times N^3$ by*

$$g = u\,\varphi^*(h) + u^{-1}\,(\mathrm{d}t + A)^2,$$

where u is a positive smooth function and A a 1-form on N^3 Then

(i) *φ is a harmonic morphism of Killing type;*

(ii) *(M^4, g) is self-dual (respectively, anti-self-dual) if u and A are related by the monopole equation*

$$\mathrm{d}u = *\mathrm{d}A \qquad \textit{(respectively, } \mathrm{d}u = -*\mathrm{d}A\textit{)}; \tag{3.5}$$

(iii) *(M^4, g) is Einstein if and only if (3.5) holds and (N^3, h) is flat, in which case g is Ricci-flat and self-dual.*

Note that the connection form θ is given by $\theta = \mathrm{d}t + A$ and $\mathrm{d}A$ is the curvature of the connection given by θ. See [35] Theorem 7.1 and [34] for more information.

Example 3.13. For $a \geq 0$, define a harmonic function on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ by

$$u_a(\boldsymbol{x}) = \frac{1}{4}\left(\frac{1}{|\boldsymbol{x}|} + a\right)\,.$$

Then the above construction gives the *Hawking Taub-Nut metric* g_a on $\mathbb{R}^4$ ($a > 0$) or the standard metric g_0 ($a = 0$), and the harmonic morphism is the Hopf polynomial map $(\mathbb{R}^4, g_a) \to (\mathbb{R}^3, h)$.

Note that the metric g_a extends to the whole of $\mathbb{R}^4$; in fact it is given by the explicit formula

$$g_a = (a|x|^2 + 1)g_0 - \frac{a(a|x|^2 + 2)}{a|x|^2 + 1}(-x_2 \mathrm{d}x_1 + x_1 \mathrm{d}x_2 - x_4 \mathrm{d}x_3 + x_3 \mathrm{d}x_4)^2.$$

See [26] for a discussion of g_1.

R. Pantilie ([32] Theorem 4.10) characterizes this example as follows:

Theorem 3.14. *Let $\varphi : M^4 \to N^3$ be a surjective harmonic morphism between complete simply-connected Einstein manifolds, which has precisely one critical point. Then up to homotheties, we have Example 3.13 for some $a \geq 0$.*

We give a second construction which produces harmonic morphisms which are, in general, neither of Killing nor of warped product type. We call these 'Type 3', see [32] for details, and see [6] §12.5 for a discussion of the general class of harmonic morphisms to which these belong.

Theorem 3.15. (cf. [9], [34]) *Let (N^3, h) be an oriented constant curvature 3-manifold and let A be a 1-form on N^3. Define a Riemannian metric on $(0, \infty) \times N^3$ by*

$$g = \rho\, h + \rho^{-1}(\mathrm{d}\rho + A)^2 \qquad (\rho \in (0,\infty))\,. \tag{3.6}$$

Then (i) *φ is a harmonic morphism.*

(ii) [35] *g is self-dual (respectively, anti-self-dual) if the following Beltrami fields equation holds on N^3:*

$$\mathrm{d}A = -*A \qquad \textit{(respectively, } \mathrm{d}A = *A\textit{)}\,. \tag{3.7}$$

(iii) [34] *g is Einstein if and only if* (3.7) *holds and h has constant curvature equal to* $1/4$, *in which case g is Ricci-flat and self-dual.*

Example 3.16. Set $U = S^3$ and $A = \alpha\, i^*(-x^2 \mathrm{d}x^1 + x^1 \mathrm{d}x^2 - x^4 \mathrm{d}x^3 + x^3 \mathrm{d}x^4)$, where $\alpha \in \mathbb{R}$ and $i : S^3 \hookrightarrow \mathbb{R}^4$ is the standard inclusion map. Then A satisfies the Beltrami fields equation, g is the Eguchi–Hansen metric ($\alpha \neq 0$) on $\mathbb{R}^4$ or the standard metric ($\alpha = 0$), and the harmonic morphism is radial projection $(\mathbb{R}^4 \setminus \{\mathbf{0}\}, g) \to (S^3, h)$, $\boldsymbol{x} \mapsto \boldsymbol{x}/|\boldsymbol{x}|$.

Note that R. Pantilie ([32] Theorem 1.8) shows that any harmonic morphism from an Einstein 4-manifold to a Riemannian 3-manifold must be of Killing type, warped-product type, or is, up to homotheties, as described in Theorem 3.15(iii). For maps from self-dual manifolds, a fourth type of harmonic morphism appears, see [35] Corollary 6.4.

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