Nonparametric density estimation on homogeneous spaces in high level image analysis

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1 Summary

The landmark data reduction approach in high level image analysis has led to significant progress to scene recognition via statistical shape analysis (Dryden and Mardia, 1998). While a number of families of similarity shape densities have proven useful in data analysis, only a few parametric models have been considered only recently in the context of projective shape (Mardia and Patrangenaru, 2004), or affine shape. Shape spaces of interest have the geometric structure of symmetric spaces: planar similarity shape spaces are complex projective spaces (Kendall, 1984), affine shape spaces are real Grassmann manifolds (Sparr, 1992), and spaces of planar projective shapes of configurations of points in general position are products of real projective spaces (Mardia and Patrangenaru, 2004). Therefore, data driven density estimation of shapes, regarded as points on symmetric spaces and arising from digitizing landmarks in images, is necessary. Recently, Pelletier (2004) considered kernel density estimation on "general" Riemannian manifolds; his results however hold only in *homogeneous spaces*. This is sufficient for image analysis, since any symmetric space is homogeneous. Pelletier estimators generalize the density estimators on certain homogeneous spaces introduced by Ruymgaart (1989), by H. Hendriks, J. H. M. Janssen and Ruymgaart (1993), and by Lee and Ruymgaart (1998). In this paper, we propose a class of adjusted Pelletier density estimators, on homogeneous spaces, that converge uniformly and almost surely at the same rate as naive kernel density estimators on Euclidean spaces.

A concrete example of projective shape density estimation of 6-ads arising from digitized images of the "actor" data set in Wayne et.al. (2001).

2 Pelletier density estimators on homogeneous spaces

A kernel density estimator of a probability distribution on an arbitrary complete Riemannian manifold (M, g) of dimension d was introduced in (Pelletier, 2004). Assume d_g is the *geodesic distance* associated with the Riemannian structure g and let $\mathcal{B}$ denote the Borel $\sigma-$field of M. Let $(\Omega, \mathcal{F}, \mathbf{P})$ denote an underlying probability space and let X be a random object on M, and let Q be the probability measure on M associated with X. If "dV", is the volume element associated with the invariant Riemannian metric g on M and Q has a probability density f w.r.t. the volume measure dV, Pelletier defines the density estimator $f_{n,K}$ of f as follows: let $K : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative map with support in $[0, 1]$ s.t. $K(\|x\|)$ is a density with finite moments of order $p \leq 2$. Then $f_{n,K}$ is defined by

$$f_{n,K}(p) = \frac{1}{n}\sum_{i=1}^{n} |g(p)|^{-\frac{1}{2}} \frac{1}{r^d} K(\frac{d_g(p, x_i)}{r}), \tag{1}$$

where $g(p)$ is the determinant of the metric, and r is the radius considered above.
Equation (1) needs to be used with caution since the coefficients of the Riemannian metric

depend on the choice of a coordinate system around p, therefore. If in addition we assume that the (M, g) is homogeneous, then for any pair of indices i, j there is an isometry $h : M \to M$, with $h(x_i) = x_j$, which insures that $f_{n,K}$ is a density.
REMARK 2.1. If M is an arbitrary compact Riemannian manifold, one may modify equation (1) as follows and get a probability density function: for each $i = 1, ..., n$ one should consider $g_i(p)$ instead of $g(p)$, where g_i is the determinant of g w.r.t. the the log-chart centered at the observation x_i. The log-chart is the inverse of the *exponential map* at x_i (Bhattacharya and Patrangenaru, 2004). To assess the quality of this estimator we shall consider the mean square error (MSE)

$$MSE(x) = \mathbf{E}\{f_{n,K}(x) - f(x)\}^2 = Var(f_{n,K})(x) + \{f_{n,r}(x) - f(x)\}^2, \tag{2}$$

and the mean integrated squared error (MISE)

$$MISE = \int_{\mathbf{M}} MSE(x)dV(x). \tag{3}$$

The following result due to Pelletier (2004) extends a result of Ruymgaart(2004):
THEOREM 2.1. *If f is of class $\mathcal{C}^2$ on M and $f_{n,K}$ is the density estimator in* (1) *, then $MISE \leq C_f(\frac{1}{nr^d} + r^4)$. Consequently for $r = O(n^{-\frac{1}{d+4}})$ we have $MISE = O(n^{-\frac{4}{d+4}})$*

3 Density estimation on symmetric spaces

If $\mathbf{M}$ is a simply connected compact symmetric space, equipped with a $\mathbf{G}-$invariant measure, it is known that M factors as a product irreducible symmetric spaces. Therefore any symmetric space $\mathbf{M}$ is locally isometric to a direct product $\mathbf{M_1} \times ... \times \mathbf{M_q}$ of irreducible symmetric spaces. Assume $p = (p_1, ..., p_q)$ is a fixed point on such a product. A vector u tangent to $\mathbf{M}$ at p can be represented as $u = (u_1, ..., u_q), u_a \in T_{p_a}M_a \forall a = 1, ..., q$, and with these identifications, the exponential map at p is given by $Exp_p(u) = (Exp_{p_1}(u_1), ..., Exp_{p_q}(u_q))$. Since exponential maps are easy to compute in irreducible symmetric spaces, for such products, rather than using caps $C_r(p)$, it is more convenient to use products of caps $C_{r_a}(p_a)$ in the irreducible factors $M_a, a = 1, ..., q$. Also, instead of using the density estimator in (1) , we use estimator $\widehat{f}_{n,K}(p)$ that is compatible with the decomposition in irreducible factors. If the random sample is $X = (X_1, ..., X_n)$, where $X_i = (X_{i1}, ..., X_{iq}), \forall i = 1, ..., n$, and $p = (p_1, ..., p_q)$ is an arbitrary point as above then

$$\widehat{f}_{n,K}(p) = \frac{1}{n}\sum_{i=1}^{n} \Pi_{a=1}^{q} |g_{ia}(p_a)|^{-\frac{1}{2}} \frac{1}{r_a^{d_a}} K(\frac{d_{g_a}(p_a, x_{ia})}{r_a}). \tag{1}$$

In (1) , $|g_{ia}(p_a)|$ is the determinant of the metric tensor of M_a at $Log_{x_{ia}}p_a$, and d_a is the dimension of M_a. For convenience, one may take equal radii $r_a, a = 1, ..., q$. The density estimator $\widehat{f}_{n,K}(p)$ has the same asymptotic order of error as $f_{n,K}(p)$ mentioned in Theorem 2.1.

4 An example of projective shape density estimation

Mardia and Patrangenaru (2004) have shown that the projective shape space $P\Sigma_m^k$, of *projective k–ads* with first $m + 2$ points of the k-ad in general position in P^m is a manifold diffeomorphic with a direct product of $q = k - m - 2$ copies of $(P^m)^q$. Using the projective frame approach

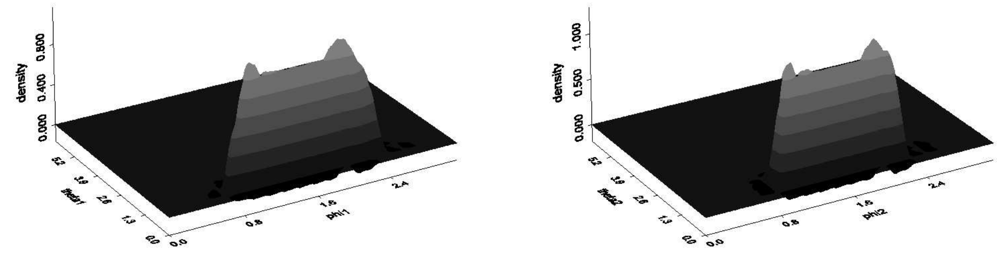

Figure 1: Actor data: marginal densities of kernel estimates in spherical coordinates

they considered the so called *spherical representation* of a shape of such a k-ad. When a distribution on the $P\Sigma_m^k$ is concentrated, one may simply regard it as a distribution on a product of q unit spheres S^m. The planar case (m=2), is one relevant in high level image analysis. In this particular case $q = k - 4$. Caution should be taken in the selection of landmarks: while the actual physical landmarks of the scene pictured are in $3D$, one should select a group of landmarks that are approximately coplanar.
The “actor” library is a data set of images of an individual who appears in different disguises in front of a camera. Sixteen such images have been digitized and six approximately coplanar landmarks (ends of eyes and lips) have been used in a planar projective shape density estimation example, using spherical caps (Ruymgaart, 2004) where K is a step function . Figure 1 displays graphs of smoothed histogram estimators of type $f_{n,r}$ for the distribution of spherical coordinates, of the the two spherical marginals associated with the 6-ads, is given by

$$f_{n,r}(\phi,\theta) = \frac{1}{nc_n(r)} \sum_{i=1}^{n} 1_{B_r(\phi,\theta)}(X_i), x \in \mathbb{R}^\mathbf{2} \tag{1}$$

where $B_r(\phi,\theta)$ is small disc of area $c_n(r)$ with center $(\phi,\theta) \in \mathbb{R}^\mathbf{2}$ and radius (bandwidth) r.

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References

Bhattacharya,R.N. and Patrangenaru,V. (2004) Large sample theory of intrinsic and extrinsic sample means on manifolds: Part II. To appear in *Ann. Statist.*

Dryden,I.L. and Mardia,K.V. (1998) *Statistical Shape Analysis* Chichester, Wiley.

Hendriks, H. , Janssen J. H. M. and Ruymgaart F. H. (1993) A Cramér-Rao type inequality for random variables in Euclidean manifolds. *Sankhya* Ser. A 54, no. 3, 387–401.

Kendall, D.G. (1984) Shape manifolds, Procrustean metrics and complex projective spaces. *Bull. London Math. Soc.* **16**, 81-121.

Kent,J.T. (1992). New directions in shape analysis. *The Art of Statistical Science, A Tribute to G.S. Watson* (K.V. Mardia, editor), Wiley, New York, 115-128.

Lee, J. M. and Ruymgaart, F. H. (1996). Nonparametric curve estimation on Stiefel manifolds. *J. Nonparametr. Statist.* 6 , no. 1, 57–68. (1998).

Mardia, K.V. and Patrangenaru, V (2004). Directions and projective shapes, tentatively accepted at *Ann. Statist.*

Pelletier, B. (2004). Kernel Density estimation on Riemannian manifolds, to appear in Statist. and Probability Letters.

Ruymgaart, F. H.(1989). Strong uniform convergence of density estimators on spheres. *J. Statist. Plann. Inference* 23 , no. 1, 45–52. (1989)

Ruymgaart, F. H.(2004). Statistics on Products of Spheres, unpublished manuscript.

Sparr, G. (1992). Depth-computations from polihedral images. *Image and Vision Computing*, **10**, 683-688.

B. Waine, K.V. Mardia, R.G. Aykroyd and J.L. Bowie(2001) Face analysis and the bivariate wrapped Cauchy distribution. *in* “Functional and Spatial Data Analysis”. *Proceedings of the 20th LASR Workshop*, edited by K.V. Mardia& R.G. Aykroyd , 55-62, Leeds University Press.